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On the Orthocomplementation of State–Property–Systems of Contextual Systems

Bart D'Hooghe

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Abstract We adopt an operational approach to quantum mechanics in which a physical system is defined by the mathematical structure of its set of states and properties. We present a model in which the maximal change of state of the system due to interaction with the measurement context is controlled by a parameter which corresponds with the number N of possible outcomes in an experiment. In the case $N = 2$ the system reduces to a model for the spin measurements on a quantum spin-1/2 particle. In the limit $N \to \infty$ the system is classical, i.e. the experiments are deterministic and its set of properties is a Boolean lattice. For intermediate situations the change of state due to measurement is neither 'maximal' (i.e. quantum) nor 'zero' (i.e. classical). We show that two of the axioms used in Piron's representation theorem for quantum mechanics are violated, namely the covering law and weak modularity. Next, we discuss a modified version of the model for which it is even impossible to define an orthocomplementation on the set of properties. Another interesting feature for the intermediate situations of this model is that the probability of a state transition in general not only depends on the two states involved, but also on the measurement context which induces the state transition.

Keywords Foundations of quantum mechanics · Operational approach · Orthocomplementation

1 Introduction

In this paper we adopt an operational approach to the foundations of quantum mechanics in which a physical system is determined by the mathematical structure of its set of states and properties, and the relation between these two sets. In the State–Property–System formalism (SPS) entities are assumed at each instant of time to be in a definite state such that some properties are *actual* in a specific state, while other properties are only *potential*, i.e. not

B. D'Hooghe (✉)
Center Leo Apostel (Clea–FUND) and Department of Mathematics, Vrije Universiteit Brussel (VUB), Brussel, Belgium
e-mail: bdhooghe@vub.ac.be

actual. Only by a change of state (induced either by measurement interaction or by free evolution) can potential properties become actual, i.e. are they *'realized'* by evolution of the (state of the) system. The SPS-formalism fits in a long history of operational approach to quantum mechanics, known as the Geneva-Brussels approach [1–3, 13, 17–19]. However, in the SPS formalism states are considered as primary concepts, and although each state can be represented in the set of properties by its property state, this state identification only follows if two conditions hold on the set of states (equivalent to separation axioms T_0 and T_1 in topology) [6]. If a specific set of quantum axioms (from Piron's Representation Theorem [18, 19], together with Solèr's additional condition [20]) is satisfied by the set of properties, it fits in a quantum mechanical representation in Hilbert space. In 1981 it was shown that a compound system consisting of two separated quantum systems does not fit structurally within the mathematical framework of standard quantum mechanics [1]. This follows from the fact that the lattice of properties of such a compound system does not obey all axioms of standard quantum mechanics, suggesting that more general mathematical structures than Hilbert spaces need to be explored in order to represent such systems. More specifically, the axioms of weak modularity and covering law are violated.

Following this operational approach, it is possible to consider a macroscopic sphere model which 'behaves quantum-like', i.e. its SPS is the same SPS as for a quantum spin-1/2 particle and its spin properties [4]. In this model, quantum probability can be explained as due to a lack of knowledge on the measurement interaction. In next step, one constructs sphere models in which a parameter controls this lack of knowledge, allowing for a continuous transition from quantum to classical systems. For the intermediate situations of the model, the axioms of weak modularity and covering law are violated [7]. Although this model is deterministic in its classical limit (no lack of knowledge on the measurement interaction), the change of state due to the measurement process is in general non-zero. In this paper we present a model in which the maximal change of state of the system due to interaction with the measurement context is controlled by a parameter which corresponds with the number N of possible outcomes in an experiment. In the case $N = 2$ the system reduces to the quantum sphere model. In the limit $N \to \infty$ the system is 'genuinely' classical, i.e. the experiments are deterministic, the set of properties is a Boolean lattice and the change of state due to measurement is zero (i.e. negligible). For intermediate situations the change of state due to measurement is neither 'maximal' (i.e. quantum) nor 'zero' (i.e. classical) and we show that the two 'suspicious' axioms of weak modularity and the covering law are violated. Next, we discuss a modified version of the model for which even the orthocomplementation on the set of properties becomes problematic.

2 An Operational Approach to Quantum Mechanics

We adopt an operational approach to quantum mechanics in which a physical entity is described by its set of states, its set of properties and a relation of 'actuality' between these two sets which expresses which properties are actual when the system is in a specific state, as follows [9]. First, we consider that at any moment the entity S is in a (known or unknown to the observer) state $p \in \Sigma$. Also, S has a set of properties $\mathcal{L}$, defined by the set of available experiments which can be performed on S. A property a is either 'actual' or 'potential' for the entity S, which means that if the property a is actual in the state p, then each time the corresponding experiment is performed, the positive outcome is found with certainty. Between the set of states and (power)set of properties is a relation $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ of 'actuality' that maps each state $p \in \Sigma$ onto the set $\xi(p)$ of those properties that are actual in this state. Dually, one can consider the Cartan map $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$, which maps a property $a \in \mathcal{L}$ onto the set of states $\kappa(a)$ that make this property actual. Depending on the nature of the entity S, one obtains a different structure on the set of states Σ, the set of properties $\mathcal{L}$ and the relation between these two sets. Hence, if we are only concerned with the structural behavior of the entity, we can focus on the triple $(\Sigma, \mathcal{L}, \xi)$. More abstractly, even without an underlying physical entity S, we can consider any two sets Σ and $\mathcal{L}$ and a function $\xi : \Sigma \to \mathcal{P}(\mathcal{L}) : p \to \xi(p)$ and study the emerging structure. The triple $(\Sigma, \mathcal{L}, \xi)$ is called a State–Property–System (SPS). The SPS-formalism is a further elaboration of the original Geneva-Brussels approach, [1–3, 13, 17–19] in which the set of experiments defines the set of properties by yes–no tests. Each physical property is identified by its set of eigenstates, i.e. if the system S is in an eigenstate of this property, the measurement yields the corresponding outcome with certainty.

If one considers the SPS of a quantum entity, one observes that certain 'quantum axioms' hold. Conversely, one can start from a general SPS, and by imposing a suitable set of axioms (the 'quantum axioms' from Piron's Representation Theorem [18, 19], with Solèr's additional condition [20]) the structure on the set of properties is such that it fits into a quantum mechanical representation. Namely, the lattice of properties is isomorphic with a family of lattices (with superselection rules) of closed subspaces in a Hilbert space over the field of reals, complex numbers or the division ring of quaternions. Let us briefly recall the axioms used in (generalized) Piron's representation theorem and more particularly at what stage the orthocomplementation enters the discussion.

Consider a SPS $(\Sigma, \mathcal{L}, \xi)$. For properties $a, b \in \mathcal{L}$ we can define an implication relation: $a < b \Leftrightarrow \kappa(a) \subset \kappa(b)$. Similarly, for states $p, q \in \Sigma$ we introduce $p < q \Leftrightarrow \xi(q) \subset \xi(p)$. It can be shown that $(\Sigma, \leq)$ and $(\mathcal{L}, \leq)$ are pre-ordered sets.

Axiom 2.1 (Property determination) *Consider a* SPS $(\Sigma, \mathcal{L}, \xi)$. *The axiom of property determination is satisfied iff for* $a, b \in \mathcal{L} : \kappa(a) = \kappa(b) \Rightarrow a = b$.

If the axiom of property determination is satisfied, then $\leq$ is a partial order relation on $\mathcal{L}$.

Axiom 2.2 (Property completeness) *The axiom of property completeness is satisfied iff* $\exists$ *generating subset* $\mathcal{T} \subseteq \mathcal{L}$ *such that* $\forall (a_i)_i \subseteq \mathcal{T}, \exists a \in \mathcal{L} : \kappa(a) = \bigcap_i \kappa(a_i)$ (*) *and* $\forall a \in \mathcal{L} : \exists (a_i)_i \subseteq \mathcal{T}$ *such that* (*) *is satisfied.*

The property a is called the *meet* of $(a_i)_i$, denoted as $a = \bigwedge_i a_i$. Consider a State–Property–System SPS $(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination and property completeness are satisfied. Then $(\mathcal{L}, <, \wedge, \vee)$ is a complete lattice. Therefore, alternatively, the combination of axioms of *property determination* and *property completeness* can be replaced by the single axiom of *completeness* of $\mathcal{L}$. For a state $p \in \Sigma$ the *property state* is defined as the property $s(p) = \bigwedge_{a \in \xi(p)} a$. The element $b \in \mathcal{L}$ is called an *atom iff* $\forall x \in \mathcal{L} : 0 < x < b \Rightarrow x = 0$ or $x = b$, i.e. b covers 0.

Axiom 2.3 (Atomicity) *Consider a* SPS $(\Sigma, \mathcal{L}, \xi)$ *for which axioms of property determination and completeness are satisfied. The axiom of atomicity is satisfied iff* $\forall p \in \Sigma : s(p)$ *is an atom of* $\mathcal{L}$ *and* $\forall a \in \mathcal{L} : a = \bigvee_{a \in \xi(p)} s(p)$.

Axiom 2.4 (Orthocomplementation) *The axiom of orthocomplementation is satisfied iff there exists an orthocomplementation relation* $^\perp : \mathcal{L} \to \mathcal{L}$ *such that for* $a, b \in \mathcal{L}$: (i) $(a^\perp)^\perp = a$, (ii) $a \leq b \Rightarrow b^\perp \leq a^\perp$, (iii) $a \wedge a^\perp = 0$ and $a \vee a^\perp = 1$.

Consider a SPS $(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination, property completeness and orthocomplementation are satisfied. Then $(\mathcal{L}, \leq, \wedge, \vee)$ is a complete orthocomplemented lattice.

Axiom 2.5 (Covering law) *The covering law is satisfied iff for* $a, b \in \mathcal{L}$ *and* $p \in \Sigma : a \wedge s(p) = 0 : a < b < a \vee s(p) \Rightarrow b = a$ or $b = a \vee s(p)$.

Axiom 2.6 (Weak modularity) *The lattice* $\mathcal{L}$ *is weakly modular iff* $\forall a, b \in \mathcal{L} : a < b \Rightarrow (b \wedge a^{\perp}) \vee a = b$.

Axiom 2.7 (Plane transitivity) *An orthocomplemented lattice* $\mathcal{L}$ *is plane transitive iff for atoms* $s, t \in \mathcal{L}$, *there are two distinct atoms* $s_1 \neq s_2$ *and a symmetry* f *such that* $f|_{[0, s_1 \vee s_2]}$ *is the identity and* $f(s) = t$.

The SPS $(\Sigma, \mathcal{L}, \xi)$ of quantum and classical mechanical entities is such that $\mathcal{L}$ is a complete orthocomplemented lattice that satisfies the covering law, and is weakly modular and plane transitive. This can be checked straightforward from the Hilbert space representation of $\mathcal{L}$ by the lattice of closed subspaces in Hilbert space for quantum systems, or the phase space representation for classical mechanical entities. The reverse statement is less trivial to show. In fact, the (re)formulation of a decisive set of axioms which forces a quantum structure on the set of properties (i.e. an isomorphism with the closed subspaces in a complex Hilbert space) has been at the heart of on-going scientific research over the last decades [18–20]. One can adopt an alternative to Solèr's axiom, namely the operational defined axiom of (6) *plane transitivity* [8], which, together with the previous five axioms, (1) *completeness*, (2) *atomicity*, (3) *orthocomplementation*, (4) *covering law*, (5) *weak modularity*, yields a full axiomatization of standard quantum mechanics:

Theorem 2.8 (Representation theorem of Piron–Solèr) *Consider a* SPS $(\Sigma, \mathcal{L}, \xi)$ *such that* $\mathcal{L}$ *contains at least* 4 *atoms and is a complete orthocomplemented atomistic lattice that satisfies the covering law, is weakly modular and plane transitive. Then* $\mathcal{L}$ *is isomorphic with a family of lattices* (*with superselection rules*) *of closed subspaces in a Hilbert space over the field of reals*, *complex numbers or the division ring of quaternions*.

3 Closure Structures

To tackle the problem of orthocomplementation on the set of properties, we represent the set of properties via an eigenclosure operation on the set of states. Introducing an orthogonality relation on the set of states, one can define an orthoclosure structure which is orthocomplemented under this orthogonality relation. Hence if these two closure structures coincide, one automatically obtains that it is possible to define an orthocomplementation on the set of properties. Let us start by showing how we can construct the SPS of a (physical) system via a closure operation on the set of states, defined in terms of eigenstate sets of experiments. We refer to [5] for a more detailed discussion of these topics, including proofs of theorems and results mentioned in next subsections.

3.1 Eigenstate Sets and Eigenclosure

Let us consider an experiment e and a subset A of the set of outcomes O_e. This defines a property a_e^A which is actual whenever the measurement e yields with certainty an outcome in A. By $eig_e(A)$ we denote the set of states for which the experiment e would yield with certainty an outcome in A, and call these the *eigenstates* of property a_e^A. Obviously, the set $eig_e(A)$ coincides with the Cartan image of the property $a_e^A : \kappa(a_e^A) = eig_e(A)$ [5]. This isomorphism justifies identifying an eigenstate set with its corresponding property. Therefore, we can study the structure of the set of properties in the state space of the entity. The following trivial properties are defined: the property $a_e^{O_e}$, which is actual in any state, and the property $a_e^{\emptyset}$, which is never actual. For the eigenstate map $eig_e : \mathcal{P}(O_e) \to \mathcal{P}(\Sigma)$, the following holds: (i) $eig_e(\emptyset) = \emptyset$, (ii) $eig_e(O_e) = \Sigma$, (iii) $\forall A_i \subset O_e : eig_e(\bigcap_i A_i) = \bigcap_i eig_e(A_i)$.

Next, let us consider a set E of experiments. A natural way to 'combine' the experiments of E is the *union experiment* e_E which consists in choosing at random an element of E and performing that experiment, and attributing the observed outcome to the experiment e_E. The outcome set of the experiment e_E is given by $O_E = \bigcup_{e \in E} O_e$. The experiments in the set E are called *primitive experiments* and the 'combination' experiment e_E is called a *union experiment*. For a set of outcomes $A \subseteq O_E$ the eigenstate set $eig_E(A)$ is defined as the set of states for which E, and therefore every experiment $e \in E$, would yield an outcome in A with certainty. The eigenstate set of a union experiment is completely defined by the eigenstate sets of the primitive experiments of which it is constructed [5]: $eig_E(A) = \bigcap_{e \in E} eig_e(A \cap O_e)$ (3.1).

One could even consider the set $\mathcal{E}$ containing all primitive experiments to construct a union experiment $e_{\mathcal{E}}$ with the outcome set $O_{\mathcal{E}} = \bigcup_{e \in \mathcal{E}} O_e$. Clearly, this outcome set $O_{\mathcal{E}}$ contains all possible experimental outcomes from all primitive and union experiments. Let us denote the collection of eigenstate sets for the union experiment $e_{\mathcal{E}}$ by $\mathcal{F}_{\mathcal{E}}$. For any union experiment E one can show that its collection $\mathcal{F}_E$ of eigenstate sets is contained in the collection of eigenstate sets $\mathcal{F}_{\mathcal{E}}$ of the union experiment $e_{\mathcal{E}}$. Therefore one can study the structure of the set of properties of the system on the basis of the structure generated by the elements of $\mathcal{F}_{\mathcal{E}}$. The following holds: $\emptyset, \Sigma \in \mathcal{F}_{\mathcal{E}}$ and $F_i \in \mathcal{F}_{\mathcal{E}}, \forall i \Rightarrow \bigcap_i F_i \in \mathcal{F}_{\mathcal{E}}$. This defines a closure operation on Σ, as follows. A *closure structure* is a couple (Σ, cl) with Σ a set and cl a mapping of $\mathcal{P}(\Sigma)$ onto itself with the following four properties: (i) $K \subseteq cl(K)$, (ii) $K \subseteq L \Rightarrow cl(K) \subseteq cl(L)$, (iii) $cl(cl(K)) = cl(K)$, (iv) $cl(\emptyset) = \emptyset$. If (X, cl) is a closure structure, then a subset F of X is called a *closed set* iff $cl(F) = F$. In the case that the set X is fixed, we can identify the closure structure (X, cl) by its set of closed sets, i.e. $\mathcal{F}_{cl} = \{F \in \mathcal{P}(X) \mid cl(F) = F\}$, and call $\mathcal{F}_{cl}$ the closure structure. This definition of a closure operation is less restrictive than the usual topological definition, for which also the (set-theoretic) union of two closed sets has to be closed. This is not necessarily the case for the closure as we define it here. The following holds: For a closure structure (X, cl), the family $\mathcal{F}$ of closed subsets has the following properties: (i) $\emptyset \in \mathcal{F}$, $X \in \mathcal{F}$, (ii) $F_i \in \mathcal{F} \Rightarrow \bigcap_i F_i \in \mathcal{F}$. On the other hand: If a family $\mathcal{F}$ of subsets of a set X is such that (i) and (ii) are satisfied, then the map $cl : \mathcal{P}(X) \to \mathcal{P}(X) : K \to cl(K) = \bigcap_{K \subseteq F_i, F_i \in \mathcal{F}} F_i$ defines a closure structure (X, cl). Since $\mathcal{F}_{\mathcal{E}}$ satisfies these conditions, this defines a closure on Σ. For an entity with state space Σ, a set of primitive experiments $\mathcal{E}$ and a collection of eigenstate sets $\mathcal{F}_{\mathcal{E}}$, the eigenclosure cl_{eig} is defined as the map $\mathcal{P}(\Sigma) \to \mathcal{P}(\Sigma)$ such that for $K \subseteq \Sigma : cl_{eig}(K) = \bigcap_{K \subseteq F_i, F_i \in \mathcal{F}_{\mathcal{E}}} F_i$. The physical interpretation of the eigenclosure is as follows. Consider a set of states A, then $cl_{eig}(A)$ is the intersection of all eigenstate sets containing the set A. This corresponds with the meet of the corresponding properties. It is the smallest possible eigenstate set containing A which corresponds with the 'smallest' possible property which is actual in all states in set A.

Let us recall that the eigenstate set for a union experiment is given by the intersection of the eigenstate sets of the experiments in the union (3.1). Therefore the collection $\bigcup_{e \in \mathcal{E}} \mathcal{F}_e$ of eigenstate sets of the primitive experiments $\mathcal{E}$ is a generating set for the collection $\mathcal{F}_{\mathcal{E}}$ of all eigenstate sets for all experiments of the entity. It is interesting to note that one could even go a step further, namely for each $e \in \mathcal{E}$ only a subset of $\mathcal{F}_e$ is required to reconstruct the whole set of eigenclosed sets $\mathcal{F}_e$ and hence also $\mathcal{F}_\mathcal{E}$.

Definition 3.1 (Generating set for a closure structure) Let (X, cl) be a closure structure and $\mathcal{F}$ the set containing all closed subsets with respect to this closure cl. The collection $\mathcal{B} \subset \mathcal{F}$ is called a *generating set* for $\mathcal{F}$ iff for every $F \in \mathcal{F}$ there exists a family $\{B_i\} \in \mathcal{B}$ such that $F = \bigcap_i B_i$.

In such case, one can show for every $K \subset X$ that $cl(K) = \bigcap_{K \subseteq B_i, B_i \in \mathcal{B}} B_i$.

Theorem 3.1 *If the axiom of property determination is satisfied, the collection* $\{eig_e(\{x_e^i\}^C) \mid e \in \mathcal{E}, x_e^i \in O_e\}$ *is a generating set for* $\mathcal{F}_\mathcal{E}$.

Proof Let $F \in \mathcal{F}_e$, i.e. $\exists A \subseteq O_e : F = eig_e(A) = \kappa(a_e^A)$. Next, let us consider an outcome $x_e^i \in A^C$ for the experiment e. Let us denote by $\{x_e^i\}^C$ the set $O_e \backslash \{x_e^i\}$. Then the property $a_e^{\{x_e^i\}^C}$ is actual in state p iff the corresponding measurement e yields an outcome in $O_e \backslash \{x_e^i\}$ with certainty. Since $A^C = \bigcup_{x_e^i \in A^C} \{x_e^i\}$ it follows that

$$A = A^{CC} = \bigcap_{x_e^i \in A^C} (O_e \backslash \{x_e^i\}).$$

Hence

$$eig_e(A) = eig_e\left(\bigcap_{x_e^i \in A^C} (O_e \backslash \{x_e^i\})\right) = \bigcap_{x_e^i \in A^C} eig_e(O_e \backslash \{x_e^i\}) = \bigcap_{x_e^i \in A^C} \kappa\left(a_e^{\{x_e^i\}^C}\right)$$
$$= \kappa\left(\bigwedge_{x_e^i \in A^C} a_e^{\{x_e^i\}^C}\right).$$

From the axiom of property determination it follows that $a_e^A = \bigwedge_{x_e^i \in A^C} a_e^{\{x_e^i\}^C}$. Therefore the family $\{a_e^{(x_e^i)^C} \mid x_e^i \in A^C\}$ generates the property a_e^A by conjunction, i.e. all eigenstate sets of primitive experiments e are generated by the eigenstate sets of the 'primary' primitive properties $a_e^{(x_e^i)^C}$. Since these generate the eigenclosure by means of the construction of the union experiment $e_\mathcal{E}$, it follows that the set of *primary primitive properties* $a_e^{(x_e^i)^C}$ is a generating set for the eigenclosure structure $\mathcal{F}_\mathcal{E}$. □

3.2 Orthogonality Relations and Orthoclosure

Another way to construct a closure operation is by considering an orthogonality relation on the set of states. Orthogonality relations can be introduced in many ways, using operationally defined concepts such as the possible measurement outcomes and state transitions due to measurement. In this section we show how an orthogonality relation on the set of states generates a closure operation on the set of states by means of the bi-orthogonal construction following Birkhoff [10, 11].

An *orthogonality relation* $\perp: \Sigma \to \Sigma$ is defined as a relation on the set of states which is (i) anti-reflexive: $\nexists p \in \Sigma : p \perp p$ and (ii) symmetric: $\forall p, q \in \Sigma : p \perp q \Rightarrow q \perp p$. The *set orthogonal* $K^\perp$ of an arbitrary set of states $K \subset \Sigma$ is defined as: $K^\perp = \{p \in \Sigma \mid p \perp q, \forall q \in K\}$. If the set K is a singleton, we can abbreviate the notation $\{p\}^\perp$ by $p^\perp$. A set K which is equal to its bi-orthogonal is called an orthoclosed set, i.e. K is *orthoclosed* iff $K = K^{\perp\perp}$. It can be shown that the bi-orthogonal operation indeed defines a closure operation [5], which justifies using the name *orthoclosure*. The set of orthoclosed sets is denoted by $\mathcal{F}_\perp$.

Next, let us consider the set $F^\perp$. Trivially, it is the set-theoretic union of its elements: $F^\perp = \bigcup_{p \in F^\perp} \{p\}$. Following the definition of the orthogonal of a set, the orthogonal of $F^\perp$ is given by the intersection of the corresponding state orthogonals: $(F^\perp)^\perp = \bigcap_{p\in F^\perp} \{p\}^\perp = \bigcap_{p\in F^\perp} p^\perp$ such that for an orthoclosed set F holds that: $F = F^{\perp\perp} = \bigcap_{p\in F^\perp} p^\perp$. This shows how the set of orthoclosed sets is generated by making intersection(s) of state orthogonals only.

3.3 Orthocomplementation of Orthoclosure and Eigenclosure

One can verify easily that $\perp$ defines an orthocomplementation on the set of orthoclosed sets. Therefore, if orthoclosure and eigenclosure coincide then the set of properties is orthocomplemented under $\perp$, which is one of the quantum axioms in the (generalized) Piron representation theorem. Since an orthoclosed set is generated as an intersection of state orthogonals, the orthoclosure is contained within the eigenclosure iff all state orthogonals are eigenclosed: $\mathcal{F}_\perp \subset \mathcal{F}_{eig}$ iff $cl_{eig}(p^\perp) = p^\perp, \forall p \in \Sigma$. On the other hand, the eigenclosure is generated by primary primitive properties $a_e^{\{x_e^i\}^C}$, such that a necessary and also sufficient condition for $\mathcal{F}_{eig} \subset \mathcal{F}_\perp$ is that $\forall e \in \mathcal{E}, \forall x_e^i \in O_e : \kappa(a_e^{\{x_e^i\}^C}) \in \mathcal{F}_\perp$, i.e. $\kappa(a_e^{\{x_e^i\}^C})$ is given by an intersection of state orthogonals. Therefore, if both conditions are satisfied, the set of eigenclosed sets coincides with the set of orthoclosed sets, and the set of properties is orthocomplemented under the $\perp$ relation.

4 The Symmetric N-Model

4.1 A Macroscopic Model with a Quantum Probability-Compatible Gun

In this and the following section we illustrate our approach by considering (possibly macroscopic) sphere models with measurements having more than two outcomes [12, 14]. The number of possible outcomes is given by a parameter N which also controls the maximal possible change of state due to measurement. For $N = 2$ the possible change of state is maximal and the system reduces to the sphere model for a quantum spin-$\frac{1}{2}$ particle. In the 'classical limit' $N \to \infty$ the measurement induces no state transition at all and all experiments are deterministic.

The physical entity S that we consider is a point particle P on the Bloch sphere (see Fig. 1). Hence the set of pure states is given by $\Sigma = \{p_v \mid v \in S^2\}$. The set of experiments is $\mathcal{E}(N) = \{e_u^N \mid u \in S^2\}, N : 2 \to \infty$ with e_u^N defined as follows. We consider the point u on the Poincaré sphere and its antipode $-u$, and divide the angular interval $[\theta_u, \theta_{-u}] = [0, \pi]$ by N equidistant angles $\theta_k = k\frac{\pi}{N-1}$, $k = 0, \dots, N-1$. The circle C_k is defined as the border of the spherical cap $cap(u, \theta_k)$, i.e. $C_k = \{q \in S^2 \mid \theta(u, q) = \theta_k\}$ and corresponds with the set of eigenstates of outcome $o_u^k = \cos(\frac{\pi k}{N-1})$, $k = 0, \dots, N-1$. Two consecutive circles C_k and C_{k+1} define a band B_k, $k = 0, \dots, N-2 : B_k = \{v \in S^2 \mid \theta_k \le \theta(u, v) \le \theta_{k+1}\}$. For

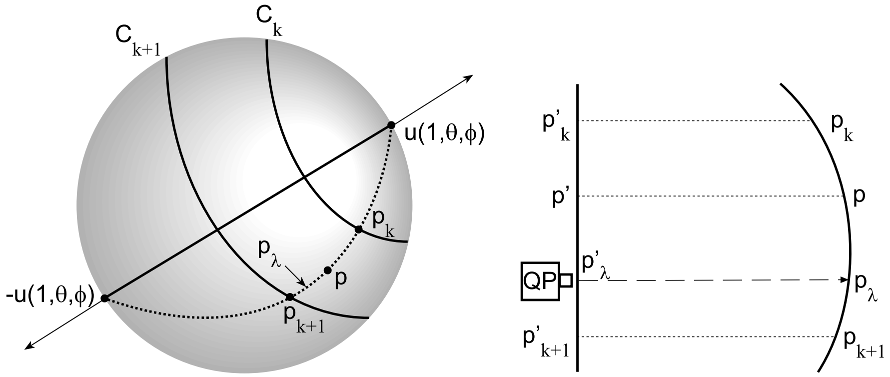

Fig. 1 The symmetric N-model and the measurement procedure with a 'quantum probability-compatible gun'

$k=0$ and $k=N-1$, the circles C_k reduce to points u and $-u$ on the sphere, respectively. The result of the measurement e_u^N is defined as follows: we consider the great circle $C_{\{p,u,-u\}}$ on S^2 through the triplet $\{p, u, -u\}$. The intersections of $C_{\{p,u,-u\}}$ with the circles C_k and C_{k+1} are denoted by p_k and p_{k+1} respectively, with orthogonal projections p'_k and p'_{k+1} onto the line segment between u and $-u$. Let us assume that the initial state p of the entity lies in the band B_k. By analogy with the ϵ-model [7], we put an elastic piece along the great circle $C_{\{p,u,-u\}}$ on S^2 between the points p_k and p_{k+1} and attach the point particle P to this elastic. The measurement process continues as the elastic breaks randomly at some point, denoted by p_λ. If $p_\lambda \in (p_{k+1}, p]$ the elastic breaks and pulls the particle P towards the point p_k, where the point particle stays attached and we assign the outcome o_u^k to the experiment e_u^N. If $p_\lambda \in (p, p_k)$ the point particle is pulled by the piece of elastic towards the point p_{k+1}, where it stays attached, and we assign the outcome o_u^{k+1} to the experiment e_u^N. The event that the elastic band breaks at exactly the point where the point particle is situated, i.e. $p_\lambda = p$, has measure zero, so our choice for the measurement procedure in this case does not affect the overall probabilities of the model.

In a sense, for the discussion of orthocomplementation of the set of properties only the set of eigenstates is relevant and the probability distributions over non-eigenstates is only of secondary importance. Nevertheless, we can always choose a procedure of 'breaking the elastic' which fits the N-model with the quantum sphere model for $N=2$. For instance, one way of breaking the elastic in a suitable way is by considering a 'quantum probability-compatible gun' moving in the interval $[-u, u]$ and shooting bullets straight at the circle segment $C_{\{p,p_k,p_{k+1}\}}$. If the gun fires when it is at a point $p'_\lambda \in [p'_k, p'_{k+1}]$, the elastic breaks at the corresponding point p_λ between p_k and p_{k+1}. If the probability that the gun fires at point p'_λ is uniformly distributed over the interval $[p'_k, p'_{k+1}]$, the resulting probability distribution over the set of outcomes coincides with the quantum probability distribution for $N=2$. Indeed, denoting the probability to obtain the outcome o_u^k if the system is in a state p as $P(o_u^k \mid p)$, one obtains that if $N=2$ the probabilities are given by $P(o_u^0 \mid p) = \frac{\cos\theta(u,p)+1}{2} = \cos^2(\frac{\theta(u,p)}{2})$ and $P(o_u^1 \mid p) = \frac{1-\cos\theta(u,p)}{2} = \sin^2(\frac{\theta(u,p)}{2})$ which are the transition probabilities for a spin measurement on a quantum spin-$\frac{1}{2}$ particle [4, 7]. See Fig. 2 for $N=5$ and $N=6$.

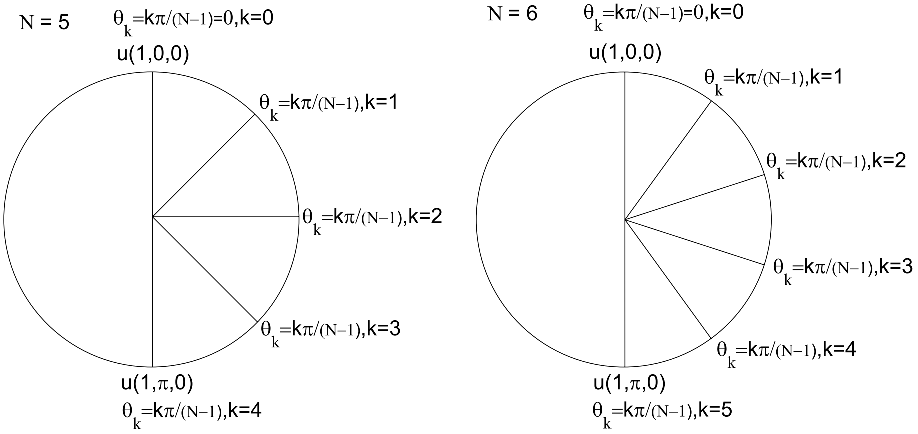

Fig. 2 The symmetric N-model for $N = 5$ and $N = 6$

4.2 Eigenclosure and Aerts Orthoclosure of the N-Model

Two states p and q are called *Aerts orthogonal* [1, 5] iff there exists a measurement $e \in \mathcal{E}$ such that p and q are eigenstates of mutually exclusive outcome sets: $p \perp_A q \Leftrightarrow \exists e \in \mathcal{E}, A, B \subset O_e, A \cap B = \emptyset : p \in eig_e(A), q \in eig_e(B)$. Hence for the N-model, a necessary condition for two states to be Aerts orthogonal is that they are separated by an angular distance at least as large as the smallest superposition angle, i.e. the smallest angle between two consecutive (non intersecting) eigenstate sets. For the symmetric N-model, these superposition angles are independent of k, since $\Delta\theta_k = \frac{\pi}{N-1}$. Hence, denoting the angle between the states p and q by $\theta(p,q)$ the necessary condition for $p \perp_A q$ is that $\theta(p,q) \geq \frac{\pi}{N-1}$. Conversely, this necessary condition is also sufficient. Let us consider states p and q such that $\theta(p,q) \geq \frac{\pi}{N-1}$. It suffices to consider the experiment e_p^N for which $p \in eig(o_u^0 = 1)$ while $q \in eig(\{o_u^0\}^C)$, which shows that states p and q are Aerts orthogonal. To summarize, denoting $\theta_{N,\perp_A} = \frac{\pi}{N-1}$ then $p \perp_A q \Leftrightarrow \theta(p,q) \geq \theta_{N,\perp_A}$.

On the other hand, the eigenclosure structure of the N-model is generated by eigenstate sets $eig_{e_u^N}(\{o_u^i\}^C), i = 0, \ldots, N-1$. Clearly, $p^{\perp_A} = \{q \in \Sigma \mid \theta(p,q) \geq \theta_{N,\perp_A}\} = eig_{e_p^N}(\{o_p^0\}^C)$ and therefore $\mathcal{F}_{\perp_A} \subset \mathcal{F}_{eig}$. Also, $eig_{e_u^N}(\{o_u^i\}^C) = \bigcap_{q \in C_i} eig_{e_q^N}(\{o_q^0\}^C) = \bigcap_{q \in C_i} q^{\perp_A}$ and hence $\mathcal{F}_{eig} \subset \mathcal{F}_{\perp_A}$. This shows that the Aerts orthoclosure structure coincides with the eigenclosure structure. Since an orthoclosure structure is orthocomplemented it follows automatically that also the lattice $\mathcal{L}$ of properties is orthocomplemented.

4.3 Failing Quantum Axioms for the Symmetric N-Model

The axioms of weak modularity and the covering law hold in the quantum case ($N = 2$) and the classical limit ($N \to \infty$). However, very similar as in the intermediate situations of the epsilon model [7, 14], these quantum axioms are violated in the intermediate cases ($2 \neq N \neq \infty$) of the symmetric N-model.

4.3.1 Weak Modularity

Recall that $p \perp_A q \Leftrightarrow \theta(p,q) \geq \theta_{N,\perp_A} = \frac{\pi}{N-1}$. Since $\mathcal{L}$ is orthocomplemented, each property can be obtained by taking meet of state property orthogonals $s(p)^{\perp_A}$ such that

$\kappa(s(p)^{\perp_A}) = cap(-p, \frac{N-2}{N-1}\pi)$. Let us choose the property a such that $\kappa(a) = cap(p, \theta_1)$ with $\frac{N-3}{N-1}\pi < \theta_1 < \frac{N-2}{N-1}\pi$, which is always possible for $N \notin \{2, \infty\}$. Property b is chosen such that $\kappa(b) = cap(p, \frac{N-2}{N-1}\pi)$. Clearly $a < b$ and $a \neq b$, such that if the axiom of weak modularity is fulfilled, there should exist a property $c \in \mathcal{L}$ orthogonal to a such that $b = a \vee c$, to be more specific $c = a^{\perp_A} \wedge b$. Since $c < a^{\perp_A}$, any state $q \in \kappa(c)$ lies in a spherical cap $cap(-p, \pi - \frac{\pi}{N-1} - \theta_1) = \bigcap_{r \in cap(p,\theta_1)} cap(-r, \frac{N-2}{N-1}\pi)$. Hence $\theta(p, q) > \pi - (\frac{N-2}{N-1}\pi - \theta_1) = \frac{\pi}{N-1} + \theta_1 > \frac{N-2}{N-1}\pi$ such that $q \notin cap(p, \frac{N-2}{N-1}\pi) = \kappa(b)$, which shows that it is impossible that $\kappa(c) \subset \kappa(b)$ and therefore $c \not< b$. Hence the axiom of weak modularity does not hold.

4.3.2 Covering Law

Next, let us consider the property a such that $\kappa(a) = cap(q, \frac{N-2}{N-1}\pi - \delta)$ with δ such that $0 < \delta < \frac{\pi}{N-1}$. Let p be a state such that $\theta(p, q) = \frac{N-2}{N-1}\pi + \delta$. Because the angle between p and q is greater than $\frac{N-2}{N-1}\pi$, the only property of which the Cartan image contains $\kappa(a)$ and $\kappa(s(p))$ is the maximal property 1 (with $\kappa(1) = \Sigma$), which shows that $a \vee s(p) = 1$. Next, take b such that $\kappa(b) = cap(q, \frac{N-2}{N-1}\pi)$. Then $a < b < a \vee s(p)$ but neither $a = b$ nor $b = a \vee s(p)$, which shows that the covering law does not hold.

5 The Asymmetric N-Model

5.1 A Macroscopic Model with Context Dependent Transition Probability

In this section we consider an asymmetric N-model in which again a parameter N controls the maximal change of state due to measurement, but with a different eigenclosure structure and different probability structure. The state of the entity is represented by a point p on the surface of the Poincaré sphere S^2. To define the experiment e_u^N we proceed as follows. We consider the point u on the Poincaré sphere and its antipode $-u$, and divide the $[u, -u]$-axis into N intervals of equal length, $I_k = [1 - \frac{2(k+1)}{N-1}, 1 - \frac{2k}{N-1}]$, $k = 0, \ldots, N-2$. The border points of the interval I_k are denoted by i_k and i_{k+1}, such that $i_k = 1 - \frac{2k}{N-1}$ and $I_k = [i_{k+1}, i_k]$, $k = 0, \ldots, N-2$. The angle θ_k is defined by $\cos\theta_k = i_k = 1 - \frac{2k}{N-1}$. The circle C_k is defined as the border of the spherical cap $cap(u, \theta_k)$, i.e. $C_k = \{q \in S^2 \mid \theta(u, q) = \theta_k\}$ and corresponds with the set of eigenstates of outcome $o_u^k = \cos\theta_k = 1 - \frac{2k}{N-1}$, $k = 0, \ldots, N-1$. Two consecutive circles C_k and C_{k+1} define a band B_k, $k = 0, \ldots, N-2 : B_k = \{v \in S^2 \mid \theta_k \leq \theta(u, v) \leq \theta_{k+1}\}$. The result of the measurement e_u^N is defined as follows: we consider the great circle $C_{\{p,u,-u\}}$ on S^2 through the triplet $\{p, u, -u\}$. The intersections of $C_{\{p,u,-u\}}$ with the circles C_k and C_{k+1} are denoted by p_k and p_{k+1}, respectively. Let us assume that the initial state of the entity is in a band B_k. The interval $[i_{k+1}, i_k]$ is divided into three pieces, by an interval $I_{k,\sup}$ of length $\frac{2}{(N-1)^2}$ centered around $m_k = \frac{i_{k+1}+i_k}{2}$, i.e. the middle of the interval I_k. Similarly to the symmetric N-model, we put an elastic piece along the great circle $C_{\{p,u,-u\}}$ on S^2 between the points p_k and p_{k+1} and attach the point particle P to this elastic. The measurement process continues as the elastic breaks randomly at some point, denoted by p_λ. If $p_\lambda \in (p_{k+1}, p]$ the elastic breaks and pulls the particle P towards the point p_k where the point particle stays attached and we assign the outcome o_u^k to the experiment e_u^N. If $p_\lambda \in (p, p_k)$ the point particle is pulled by the piece of elastic towards the point p_{k+1}, where it stays attached, and we assign the outcome o_u^{k+1} to the experiment e_u^N. Again, the event that the elastic band breaks at exactly the point where the point particle

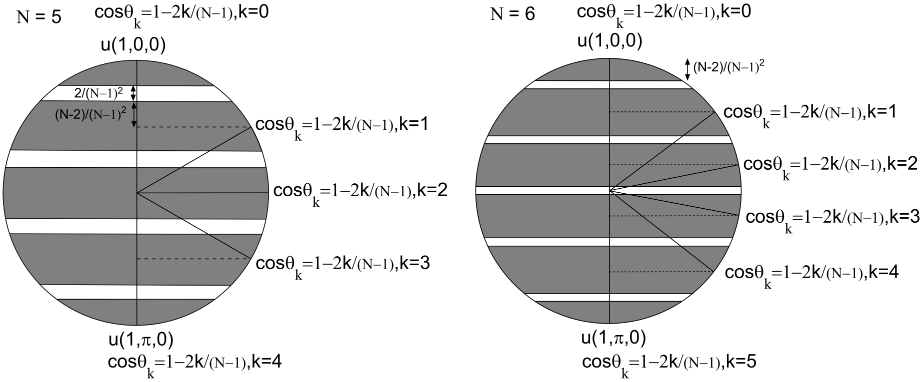

Fig. 3 The asymmetric N-model for $N = 5$ and $N = 6$, the eigenstate sets of the respective outcomes are represented by spherical sectors coloured in *gray*

is situated, i.e. $p_\lambda = p$, has measure zero, so our choice for the measurement procedure in this case does not affect the overall probabilities of our model. We choose a procedure of 'breaking the elastic' as follows. The *quantum probability-compatible gun* shoots bullets straight at the circle segment $C_{\{p,p_k,p_{k+1}\}}$, but is restricted to the interval $I_{k,\sup}$. Moreover, it is defined such that the probability that the gun fires at point p'_λ is uniformly distributed over the interval $I_{k,\sup}$. If the gun fires when it is at point $p'_\lambda \in [p'_k, p'_{k+1}]$, the elastic breaks at the corresponding point p_λ between p_k and p_{k+1}. To conclude, there are three cases: (i) $p' \in I_{k,\mathrm{up}} = [1 - \frac{2k}{N-1} - \frac{N-2}{(N-1)^2}, 1 - \frac{2k}{N-1}]$ and the elastic pulls the point particle towards p_k, where it stays attached and the measurement e_u^N yields the outcome $o_u^k = 1 - \frac{2k}{N-1}$ with certainty; (ii) $p' \in I_{k,\mathrm{down}} = [1 - \frac{2(k+1)}{N-1}, 1 - \frac{2(k+1)}{N-1} + \frac{N-2}{(N-1)^2}]$ and the elastic pulls the point particle towards p_{k+1}, where it stays attached and the measurement e_u^N yields the outcome $o_u^{k+1} = 1 - \frac{2(k+1)}{N-1}$ with certainty; or (iii) $p' \in I_{k,\sup} =]1 - \frac{2(k+1)}{N-1} + \frac{N-2}{(N-1)^2}, 1 - \frac{2k}{N-1} - \frac{N-2}{(N-1)^2}[$ we consider a random variable λ in the interval $I_{k,\sup}$ by employing the quantum probability-compatible gun. If $\lambda \le p'$ the measurement induces a state transition from p to p_k and the experiment yields the outcome $o_u^k = 1 - \frac{2k}{N-1}$; if $\lambda > p'$ the measurement induces a state transition from p to p_{k+1} and the experiment yields the outcome $o_u^{k+1} = 1 - \frac{2(k+1)}{N-1}$. See Fig. 3 for $N = 5$ and $N = 6$.

For $N = 2$ the probability distribution over the set of outcomes coincides with the quantum probability distribution, i.e. the transition probabilities for a spin measurement on a quantum spin-$\frac{1}{2}$ particle. In the limit $N \to \infty$ the system has a classical structure, i.e. its set of properties is a Boolean lattice. The orthogonal of a state p is given by the set-theoretic complement $\{p\}^C$, and the eigenstate set of property $eig_{e_p^{N\to\infty}}(\{o_p^0\}^C)$ is $\{p\}^C$. Therefore the eigenclosure coincides with the identity map such that each subset $F \subset \Sigma$ corresponds to an eigenclosed set. Hence the set of properties is isomorphic with $\mathcal{P}(\Sigma)$ and it is not difficult to see that this is indeed a Boolean (i.e. distributive) lattice. In the classical limit each measurement can be regarded as an *observation* only such that the state transition due to measurement is zero.

One of the features of this model is that the transition probability between states p and q not only depends on the two states involved, but also on the specific measurement used to induce this state transition. Let $\theta_{N,\perp_D}$ be defined as follows:

(i) for even N : $\theta_{N,\perp_D} = \arcsin(\frac{1}{N-1}) + \arcsin(\frac{1}{(N-1)^2})$

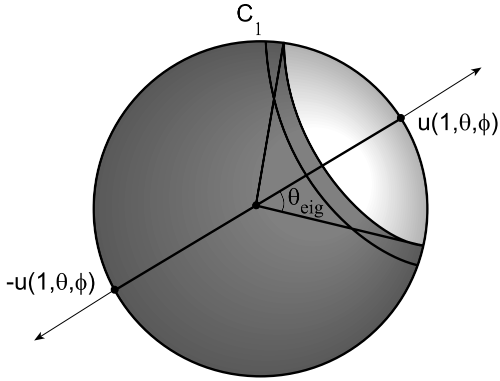

Fig. 4 The 'largest' non-trivial eigenstate set for an experiment e_u^N, namely $eig_{e_u^N}(\{o_u^0\}^C)$, which is the spherical cap $cap(-u, \pi - \theta_{eig})$ with $\cos\theta_{eig} = 1 - \frac{N}{(N-1)^2}$

(ii) for odd $N: \theta_{N,\perp_D} = \arcsin(\frac{N}{(N-1)^2})$

then if one considers p and q such that $\theta(p,q) = \theta_{N,\perp_D}$ it is possible to show there exists at least one experiment e_v^N with q as a possible final state but which does not induce a state transition from p to q, so $P_{e_v^N}(p \mid q) = 0$ (in fact, this also allows to define an orthogonality relation in terms of state transitions instead of eigenstates [12–14]). Now, let us consider the experiment e_p^N. Then one can check that $P_{e_p^N}(p \mid q) \neq 0$, showing that the state transition probability indeed depends on the measurement context which induces it. However, since the focus of the present paper lies on the quantum axiom of orthocomplementation, we will not go deeper into this issue here.

5.2 Eigenclosure and Aerts Orthoclosure of the Asymmetric N-Model

Similarly as for the symmetric N-model, one can show that states p and q are Aerts orthogonal iff they are separated by an angle greater than the smallest superposition angle. These superposition angles can be explicitly calculated as a function of k, the index of the superposition zone. We find that the minimal superposition angle is obtained for $k = \frac{N}{2}$ for even N, and $k = \frac{N-1}{2}$ for odd N. Denoting the smallest superposition angle as $\theta_{N,\perp_A}$, we obtain for even $N: \theta_{N,\perp_A} = 2\arcsin(\frac{1}{(N-1)^2})$ and for odd $N: \theta_{N,\perp_A} = \arcsin(\frac{N}{(N-1)^2}) - \arcsin(\frac{N-2}{(N-1)^2})$. Hence two states p and q are Aerts orthogonal iff $\theta(p,q) \geq \theta_{N,\perp_A}$ [12].

The 'largest' non-trivial eigenstate set for an experiment e_u^N is given by $eig_{e_u^N}(\{o_u^0\}^C)$. This is a spherical cap $cap(-u, \pi - \theta_{eig})$ with θ_{eig} given by $\cos\theta_{eig} = 1 - \frac{N}{(N-1)^2}$ (see Fig. 4).

For $2 \neq N \neq \infty$ one finds that $\theta_{eig} > \theta_{N,\perp_A}$, which means that the state orthogonals $p^{\perp_A}$ are not contained in a non-trivial eigenstate set, i.e. $p^{\perp_A}$ is not eigenclosed. Since for an orthoclosure structure the state orthogonals generate the orthoclosed sets by intersection, i.e. $F \in \mathcal{F}_\perp \Leftrightarrow F = \bigcap_{q \in F^\perp} q^\perp$, it follows immediately that the inclusion $\mathcal{F}_{\perp_A} \subseteq \mathcal{F}_{eig}$ cannot hold. The possibility of the reversed inclusion $\mathcal{F}_{eig} \subseteq \mathcal{F}_\perp$ is discussed in next subsection.

5.3 Problem of Orthocomplementation on the Set of Properties

Let us show under which conditions it is possible to define an orthogonality relation $\perp_N$ on the set of states of the asymmetric N-model such that $\mathcal{F}_{eig} = \mathcal{F}_{\perp_N}$, i.e. such that the set of eigenstate sets is orthocomplemented via this orthogonality relation. Let us assume first that the orthogonality relation $\perp_N$ depends on the angle between the states only, i.e. there exists $\theta_{\perp_N}$ such that $p \perp_N q \Leftrightarrow \theta(p,q) \geq \theta_{\perp_N}$. Let us formulate the necessary and sufficient conditions on $\theta_{\perp_N}$ such that $\mathcal{F}_{eig} = \mathcal{F}_{\perp_N}$ and sketch briefly some of the proofs to obtain the main results.

Theorem 5.1 *$\mathcal{F}_{\perp_N} \subseteq \mathcal{F}_{eig}$ iff $\theta_{\perp_N} \geq \theta_{eig}$.*

Proof For $\mathcal{F}_{\perp_N} \subseteq \mathcal{F}_{eig}$ to hold, it is necessary and sufficient that each state orthogonal is eigenclosed. Since a state orthogonal is given by $cap(-p, \pi - \theta_{\perp_N})$ the opening angle of the largest non-trivial eigenstate set (i.e. $eig_{e_p^N}(\{o_p^0\}^C)$) should be at least as large as $\pi - \theta_{\perp_N}$. Since $eig_{e_p^N}(\{o_p^0\}^C) = cap(-p, \pi - \theta_{eig})$ it is necessary that $\theta_{\perp_N} \geq \theta_{eig}$. This is also a sufficient condition: if $\theta_{\perp_N} \geq \theta_{eig}$ then each state orthogonal can be written as an intersection of eigenstate sets. This can be seen by considering the complement of the state orthogonal, i.e. $(p^{\perp_N})^C$. This is an open spherical cap $cap^\circ(p, \theta_{\perp_N})$. Since $\theta_{\perp_N} \geq \theta_{eig}$ this set can be covered by open caps $cap^\circ(q, \theta_{eig})$ such that $cap^\circ(p, \theta_{\perp_N}) = \bigcup_{q \in Q} cap^\circ(q, \theta_{eig})$ for some suitable set Q. Therefore,

$$p^{\perp_N} = \left((p^{\perp_N})^C\right)^C = (cap^\circ(p, \theta_{\perp_N}))^C = \left(\bigcup_Q cap^\circ(q, \theta_{eig})\right)^C$$

$$= \bigcap_Q cap^\circ(q, \theta_{eig})^C = \bigcap_Q cap(-q, \pi - \theta_{eig}) = \bigcap_Q eig_{e_q^N}\left(\{o_q^0\}^C\right)$$

which shows that $p^{\perp_N}$ is eigenclosed. □

For $N > 2$, let us define θ_m as the smallest angular distance between the eigenstate sets $eig_{e_u^N}(\{o_u^l \mid l = 0, \ldots, k-1\})$ and $eig_{e_u^N}(\{o_u^l \mid l = k+1, \ldots, N-1\})$. It can be shown that θ_m is given by:

$$\theta_m(N \text{ odd}) = 2 \arcsin\left(\frac{N}{(N-1)^2}\right), \tag{1}$$

$$\theta_m(N \text{ even}) = \arcsin\left(\frac{1}{(N-1)^2}\right) + \arcsin\left(\frac{2N-1}{(N-1)^2}\right). \tag{2}$$

The following theorem holds, and is illustrated on Fig. 5:

Theorem 5.2 *$\mathcal{F}_{eig} \subseteq \mathcal{F}_{\perp_N}$ iff $\theta_{\perp_N} \leq \frac{\theta_m}{2}$ and $\theta_{\perp_N} \leq \theta_{eig}$.*

Proof If $\mathcal{F}_{eig} \subseteq \mathcal{F}_{\perp_N}$ then necessarily each eigenstate set can be written as an intersection of state orthogonals. More concretely, it is necessary and sufficient that the eigenstate sets $eig_e(O_e \backslash \{1 - \frac{2k}{N-1}\}), k = 0, \ldots, N-1$ are orthoclosed, since they generate the lattice of properties $\mathcal{F}_{eig}$. Considering that the eigenstate set $eig_e(O_e \backslash \{1\})$ should be contained within a state orthogonal, one obtains immediately as necessary condition $\theta_{\perp_N} \leq \theta_{eig}$. Next, following the same procedure as in the proof of the previous theorem, we can show that the set $eig_e(O_e \backslash \{1 - \frac{2k}{N-1}\}), k = 1, \ldots, N-1$ is orthoclosed iff $\theta_{\perp_N} \leq \frac{\theta_m}{2}$. Indeed, in such a case the set-theoretic complement of $eig_e(O_e \backslash \{1 - \frac{2k}{N-1}\})$ can be written as a union of open spherical caps $cap^\circ(q, \theta_{\perp_N})$:

$$\left(eig_e\left(O_e \Big\backslash \left\{1 - \frac{2k}{N-1}\right\}\right)\right)^C = \bigcup_Q cap^\circ(q, \theta_{\perp_N}).$$

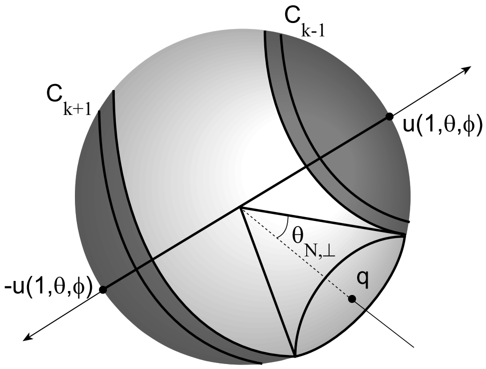

Fig. 5 Problem of orthocomplementation on the set of properties $\mathcal{L}$: Illustration why $\theta_{\perp_N} \leq \frac{\theta_m}{2}$ is necessary for $\mathcal{F}_{eig} \subseteq \mathcal{F}_{\perp_N}$

Taking the set-theoretic complement a second time, we obtain:

$$eig_e\left(O_e \Big\backslash \left\{1 - \frac{2k}{N-1}\right\}\right) = \bigcap_Q cap(-q, \pi - \theta_{\perp_N})$$

which means that $eig_e(O_e \backslash \{1 - \frac{2k}{N-1}\})$ is orthoclosed. This procedure can be repeated for all values of k, from which the necessary condition follows: $\theta_{\perp_N} \leq \frac{\theta_m}{2}$. Following a similar reasoning one can show that $\theta_{\perp_N} \leq \frac{\theta_m}{2}$ is also a sufficient condition for $eig_e(O_e \backslash \{1 - \frac{2k}{N-1}\})$, $k = 1, \ldots, N-1$ to be orthoclosed. □

Combining Theorems (5.1) and (5.2), the necessary and sufficient condition on $\theta_{\perp_N}$ to obtain $\mathcal{F}_{eig} = \mathcal{F}_{\perp_N}$ is given by:

Theorem 5.3 *$\mathcal{F}_{eig} = \mathcal{F}_{\perp_N}$ iff $\theta_{\perp_N} = \theta_{eig}$ and $\theta_{\perp_N} \leq \frac{\theta_m}{2}$.*

Using expressions (1) and (2) for θ_m, one can check whether $\theta_{eig} \leq \frac{\theta_m}{2}$—i.e. $\mathcal{F}_{eig} = \mathcal{F}_{\perp_N}$—is possible *at all*. It turns out that for all finite values of $N > 2$ the inequality $\theta_{eig} \leq \frac{\theta_m}{2}$ is violated. This shows that there cannot exist an orthogonality relation $\perp_N$ such that $\mathcal{F}_{eig} = \mathcal{F}_{\perp_N}$.

Due to Theorem (5.3), for $N > 2$ the asymmetric N-model has no orthocomplementation on the set of properties which can be defined by means of an orthogonality relation on the set of states, i.e. which is characterized by an orthogonality angle $\theta_\perp$ such that $p \perp q \Leftrightarrow \theta(p,q) \geq \theta_\perp$.

Of course, one could still imagine other 'non-standard' procedures of defining an orthocomplementation on the set of properties, i.e. without making use of the bi-orthogonal construction. Let us assume for a moment that the N-model for $N > 2$ would allow an orthocomplementation, denoted by $'$. One can show that the properties are generated by taking meets of (property) state orthocomplements, as follows. First, the set of states of the N-model is atomistic, such that each property a can be written as the join of its property states: $a = \bigvee_{s_p < a} s_p$. Also, $\kappa(s_p) = \{p\}$. Hence one can abbreviate the previous expression as $a = \bigvee_{p<a} p$. Then analogously $a' = \bigvee_{p<a'} p$, such that, using De Morgan laws, one has $a = a'' = (\bigvee_{p<a'} p)' = \bigwedge_{p<a'} p'$, which shows that the properties are generated by taking meets of (property) state orthocomplements (similar as the orthoclosure is generated by state orthogonals). On the other hand, the eigenclosure structure of the asymmetric N-model is generated by the eigenstate sets $eig_e(O_e \backslash \{1 - \frac{2k}{N-1}\})$, which—as we have seen—do not

have a single family of generators, e.g., the eigenstate set $eig_e(O_e \backslash \{1 - \frac{2k}{N-1}\}), k = \frac{N}{2}$ (or $k = \frac{N-1}{2}$ for odd N) cannot be generated by making intersections of eigenstate sets of the form $eig_e(O_e \backslash \{1\})$ and *vice versa*. This means that for $N > 2$ there exist at least two 'incompatible' generating subsets for its eigenclosure structure. Therefore it is impossible to recover the eigenclosure structure via an orthocomplemented structure which is generated by the *single* family of generators $\{\kappa(p') \mid p \in \Sigma\}$.

6 Conclusions

We have presented a model in which the change of state of the system induced by interaction with the measurement context is controlled by a parameter N reflecting the number of outcomes. In the case $N = 2$ the system reduces to a model for the spin measurements on a quantum spin-1/2 particle. In the limit $N \to \infty$ the system has a classical structure, i.e. experiments are deterministic and the set of properties is represented by a Boolean lattice. For intermediate values of the parameter, the change of state under measurement is neither maximal (i.e. quantum) nor zero (i.e. classical), and the system does not fit within a quantum Hilbert space nor a classical phase space description. To deal with these issues in a rigorous mathematical way, we have constructed the SPS of these models for different values of the contextuality parameter N and studied the problem of orthocomplementation on the set of properties. We have presented an asymmetric version of the sphere model for which it is even impossible for $N > 2$ to define an orthocomplementation on the set of properties via an orthogonality relation on the set of states. There exist at least two 'incompatible' generating subsets for the eigenclosure structure of the asymmetric N-model. Therefore it is impossible to recover the eigenclosure structure via an orthocomplemented structure which is generated by a *single* family of generators $\{\kappa(p') \mid p \in \Sigma\}$.

Another interesting feature for the intermediate situations of the asymmetric N-model (not discussed in full detail in this paper), is that the probability of a state transition in general depends not only on the (angular distance between the) two states but also on the measurement context which induces the state transition. In the axiomatic foundations of quantum theory, the theorem of Gleason dictates the uniqueness of the transition probability function between two states in Hilbert space [16]. Moreover, this probability function only depends on (the angle in Hilbert space between) the initial and the final state. Therefore Gleason's theorem does not apply to the asymmetric N-model, implying that the probability distribution over the set of outcomes cannot be derived from the structure of the set of properties in the same way as for quantum entities. In this sense our model also sheds new light on Gleason's theorem and suggests that transition probability should not be treated as a secondary concept which can be derived from other 'more basic' concepts such as states and properties as in Gleason's theorem, but instead should be regarded as a primitive concept by its own right. In this sense this macroscopic model should not only be considered as a mathematical exploration of possible generalizations of quantum mechanics *an sich*, but also as a further elaboration along the lines of the physical motivations of the original hidden measurement model given by Aerts, which aimed to provide a possible explanation for the probabilities of quantum mechanics. The current model allows to explore the (possible) link between the axiom of orthocomplementation and the measurement (in)dependence of the quantum probability distribution, in a similar way as Gisin showed how to derive the Hilbert space structure of quantum mechanics from assuming the uniqueness of the quantum probability function [15]. In a forthcoming paper, we intend to address these issues by constructing the State Context Property System for the N-models, in which measurement context is explicitly taken into account.

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